CONDITIONAL STABLE MATCHINGS

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Abstract. In matching theory of contracts the substitutes condition plays an essential role to ensure the existence of stable matchings. We study many-to-many matchings where groups of individuals, of size possibly greater than two, are matched to a set of institutions. Real-world examples include orphan brothers accepting an adoptive family conditional on all of them being included; hiring contracts that may only be chosen together; or a situation where a firm accepts to hire several workers only if they accept to work on different days (part-time jobs).

We demonstrate by several examples that such extra conditions may alter the natural choice maps so that stable matchings cannot be obtained by applying the standard theorems. We overcome this difficulty by introducing a new construction of choice maps. We prove that they yield stable matchings if the construction respects an "anti-trust" rule on the supply side of the market.

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2000 *Mathematics Subject Classification.* Primary: 91B68; Secondary: 90C27.

Key words and phrases. Primary: games, matchings, choice maps, blocs; Secondary: substitutes condition.

Acknowledgment. The work of the second author has been partially supported by a Summer Faculty Fellowship from the University of Maryland-Baltimore County.

1. Introduction

Since the pioneering work of Gale and Shapley [10] on stable matchings, their theory and algorithm has been generalized and adapted to a multitude of problems of practical interest. Following Kelso and Crawford [14] and Roth [20], many of these results may be conveniently formulated using choice functions. Several important examples are given in the monograph of Roth and Sotomayor [24].

Roth [20] provides a first analysis of group matchings, looking at the assignment of married medical students to hospitals in the same community. In his framework, each hospital offers one position and has strict preferences over students. Each couple has strict preferences on ordered pairs of hospitals. He shows that the set of stable outcomes may be empty, which he found consistent with the fact that most couples make their arrangement outside of the National Resident Matching Program. Using the same framework, Klaus and Klijn [15] show that stable matchings exist for a domain of "weakly responsive" preferences.[1] Their result is consistent with the idea that a sufficient amount of substitutability implies the existence of desirable outcomes (see, e.g., Roth [22]; Kelso and Crawford [14]; Alkan and Gale [1], Hatfield and Milgrom [13]) except that in their paper, the substitutability condition is imposed on the demand side of the market instead of being imposed on the supply side of the market. From a practical point of view, although a non-negligible proportion of the population of couples may have such preferences, the proportion of those who do may be difficult to measure: where one needs to devise a centralized matching procedure, it is often difficult to verify that agents' preferences satisfy this or that property. Moreover, this preference structure is less and less likely to be valid as we increase the group size from two members to more members. Echenique and Yenmez (2007) [7] propose a many-to-one matching algorithm, for example of students to colleges, where the students have preferences over the other students who would attend the same college. Although they do not obtain a general structure on preferences that would guarantee existence of some solution to the model, their algorithm finds the solutions, if they exist and does not require any assumption on preferences. Their algorithm is extended to a model where colleges have preferences over couples, and each student has preferences over a set of partners and colleges.

In this paper, we consider groups of individuals or firms, of size possibly greater than two, such that each firm has strict preferences over workers and has a quota on the number of workers to hire; each worker has strict preferences over firms; a group of workers may be hired by a group of firms of their choice; a group of firms may hire a group of workers of their choice. Hence, our framework extends the one to many matching procedure above to a possible many-to-many matching procedure. It is adapted, but not limited, to the following examples:

(1) In a married couple the husband accepts a position in firm F_1 only if his wife is hired by firm F_2. This example also illustrates the framework of Roth [20] or Klaus and Klijn [15].
(2) A firm hires several workers only if they accept to work on different days.

[1] A couple's preferences are responsive if the unilateral improvement of one partner's job is considered beneficial for the couple as well. If responsiveness only applies to acceptable positions, then preferences are weakly responsive

(3) Consider a married couple and a husband's ex-wife. The husband accepts a position at F_1 provided that his wife is hired at firm F_2 and ex-wife is hired at firm F_3.

Example 2 illustrates the part-time labor market. Example 3 can extend to illustrate such unraveling market as the market for children adoption where brothers and sisters may be assigned to different host families, (each with a maximum number of "potential" children) provided that families are located within a certain mile radius.

The question addressed in this paper is whether there exists a choice map that guaranties the existence of a stable assignment outcome. We demonstrate by several examples that such extra conditions may alter the natural choice maps so that stable matchings cannot be obtained by applying the standard theorems. We overcome this difficulty by introducing a new construction of choice maps. Our main result shows that there exists a stable set of contract matchings provided that firms give preferences to small groups rather than large groups. The intuition is that firms see groups of workers as more substitutes, the smaller their size. Indeed, smaller groups represent less constraints on the firms'preferences, hence increasing the overall assignment satisfaction.

The structure of the paper is the following. In the next section we briefly recall some basic notions and results in matching theory that we need for the sequel. In Section 3 we investigate the possible generalization of the known theorems to the case of conditional matching problems. In Section 4 we illustrate the usefulness of our results on a model problem.

2. General Framework

Many earlier theorems on the existence of stable matchings are special cases of an abstract theorem of Fleiner [8], [9]. We introduce the section by some definitions to be used throughout the paper. We then recall an equivalent form of Fleiner's theorem, given in [18], which we illustrate by some examples. Without loss of generality, we present our framework in the context of the job market, where firms are indexed by F and workers are indexed by W.

Definition 2.1. Given a nonempty set X, by a *choice function* on X we mean a map $C : 2^X \to 2^X$, where 2^X denotes the family of all subsets of X, satisfying the following condition:

$$C(A) \subset A \quad \text{for all} \quad A \subset X.$$

Here X represents the set of all possible contracts. If a certain set A of contracts is proposed, then the corresponding agent selects from this set a subset $C(A)$ of accepted contracts.

Definition 2.2. Given two choice functions $C_W, C_F : 2^X \to 2^X$ on the same set X, modeling two competing sides, a set $S \subset X$ is said to be *stable* if there exist two sets $S_W, S_F \subset X$ satisfying the following conditions:

$$S_W \cup S_F = X; \tag{2.1}$$

$$S \subset S_W \cap S_F; \tag{2.2}$$

$$C_W(A) = S \quad \text{for every} \quad S \subset A \subset S_W; \tag{2.3}$$

$$C_F(A) = S \quad \text{for every} \quad S \subset A \subset S_F. \tag{2.4}$$

Intuitively, for any proposed set of contracts, the set that is accepted by workers matches the set accepted by firms. Stable contract sets represent acceptable compromises.

This definition is equivalent to other usual definitions of stable matchings (see [18, Proposition 3.5]). It follows at once from the definition that a stable set is *not blocked* by any other contract, i.e., for each $b \in X$ we have either $C_W(S \cup \{x\}) = S$ or $C_F(S \cup \{x\}) = S$ (or both).

In order to ensure the existence of stable sets we need an extra property.

Definition 2.3. A choice map $C : 2^X \to 2^X$ satisfies the *strengthened substitutes condition* if

$$A, B \subset X \quad \text{and} \quad C(A) \subset B \Longrightarrow A \cap C(B) \subset C(A).$$

This condition means that if a contract is rejected from some proposed set A, then it will also be rejected from every other proposed set B which contains the accepted contracts.

It is shown in [18, Proposition 3.2] that a choice map satisfies the *strengthened substitutes condition* if and only if it is *consistent* or *satisfies the path independence property*:

$$C(A) \subset B \subset A \Longrightarrow C(B) = C(A),$$

and it satisfies the *substitutes condition*:

$$A, B \subset X \quad \text{and} \quad A \subset B \Longrightarrow A \cap C(B) \subset C(A).$$

Intuitively, workers are substitutes in their talent provided that a worker rejected from a set of potential hires is rejected regardless of the amount of other rejected workers, as long as the accepted workers remain in the set. This definition corresponds to the definition of substitutability in Roth and Sotomayor [24]; F is said to have "substituable" preferences, if any preferred set of employees (from any subset of X) that includes w remains its preferred set of employees from any subset of X that still includes w. Hence, F continues to want to employ w even if some of the other workers become unavailable.

We recall a classical construction due to Roth [23] of choice maps having this property.

Example 2.4. Given a finite subset $Y \subset X$, a nonnegative integer q (called *quota*) and a strict preference ordering $y_1 \succ y_2 \succ \cdots$ on Y, we define a map $C(A)$ for any given $A \subset X$ as follows. If $|A \cap Y| \le q$, then we set $C(A) := A \cap Y$. (Here and in the sequel we denote by $|B|$ the number of elements of a set B.) Otherwise let $C(A)$ be the set of the first q elements of $A \cap Y$ according to the ordering of Y. Then $C : 2^X \to 2^X$ is a choice map on X.

This choice map satisfies the strengthened substitutes condition: we prove a more general theorem later in Theorem 3.10: see Remark 3.11 (i).

We also recall [18, Theorem 3.6]:

Theorem 2.5. *If the choice maps $C_W, C_F : 2^X \to 2^X$ satisfy the strengthened substitutes condition, then there exists at least one stable set of contracts.*

Examples 2.6. We give two examples demonstrating the necessity of the strengthened substitutes condition.

Consider a two-point set $X = \{a, b\}$ and the three choice maps defined by the following formulae:

A	$\varnothing$	$\{a\}$	$\{b\}$	$\{a, b\}$
$C_1(A)$	$\varnothing$	$\{a\}$	$\{b\}$	$\{a\}$
$C_2(A)$	$\varnothing$	$\varnothing$	$\{b\}$	$\varnothing$
$C_3(A)$	$\varnothing$	$\varnothing$	$\varnothing$	$\{a, b\}$

One may readily verify that

- C_1 satisfies the strengthened substitutes condition,
- C_2 satisfies the substitutes condition but is not path-independent,
- C_3 is path-independent but does not satisfy the substitutes condition.

(a) If $C_W = C_1$ and $C_F = C_2$, then there is no stable set. Indeed, we have $C_W(S) = S = C_F(S)$ only if $S = \varnothing$ or $S = \{b\}$. However, $S = \varnothing$ is blocked by $\{b\}$ because

$$C_W(S \cup \{b\}) = C_F(S \cup \{b\}) = \{b\} \neq S,$$

and $S = \{b\}$ is blocked by $\{a\}$ because

$$C_W(S \cup \{a\}) = \{a\} \neq S \quad \text{and} \quad C_F(S \cup \{a\}) = \varnothing \neq S.$$

Hence none of these sets is stable.

(b) If $C_W = C_1$ and $C_F = C_3$, then there is no stable matching either. Indeed, we have $C_W(S) = S = C_F(S)$ only if $S = \varnothing$. For $S = \varnothing$ the condition (2.3) is satisfied only if $S_W = \varnothing$, and then $S_F = X$ by (2.1). However, then $C_F(S_F) = X \neq S$, so that (2.4) fails.

Remark 2.7. AygÃ¼n and SÃ¶nmez [2],[3] have recently shown that "irrelevance of rejected contracts", the condition that

$$z \notin C(Y \cup \{z\}) \Longrightarrow C(Y) = C(Y \cup \{z\}),$$

is essential for many "matching with contracts" models. Let us show that this condition follows from the strengthened substitutes condition. The case $z \in Y$ being obvious, we may assume that $z \notin Y$.

Assuming thus that $z \notin C(Y \cup \{z\})$ and $z \notin Y$, we prove the two inclusions $C(Y) \subset C(Y \cup \{z\})$ and $C(Y \cup \{z\}) \subset C(Y)$ by two different applications of the strengthened substitutes condition.

First we have

$$\begin{aligned} z \notin C(Y \cup \{z\}) &\Longrightarrow C(Y \cup \{z\}) \subset Y \\ &\Longrightarrow (Y \cup \{z\}) \cap C(Y) \subset C(Y \cup \{z\}) \\ &\Longrightarrow C(Y) \subset C(Y \cup \{z\}); \end{aligned}$$

the first implication follows from the choice map property $C(Y \cup \{z\}) \subset Y \cup \{z\}$ and from the assumption $z \notin C(Y \cup \{z\})$, the second one is the application of the strengthened substitutes condition with

$$A = Y \cup \{z\} \quad \text{and} \quad B = Y,$$

while the third one follows again from the choice map property: since $C(Y) \subset Y$, we have $(Y \cup \{z\}) \cap C(Y) = C(Y)$.

On the other hand, we also have the converse inclusion:

$$\begin{aligned} C(Y) \subset Y \cup \{z\} &\Longrightarrow Y \cap C(Y \cup \{z\}) \subset C(Y) \\ &\Longrightarrow C(Y \cup \{z\}) \subset C(Y); \end{aligned}$$

here the first implication follows by applying the strengthened substitutes condition with

$$A = Y \quad \text{and} \quad B = Y \cup \{z\},$$

while the second one follows because $Y \cap C(Y \cup \{z\}) = C(Y \cup \{z\})$ by the choice map property $C(Y \cup \{z\}) \subset Y \cup \{z\}$ and by our assumption $z \notin C(Y \cup \{z\})$.

Example 2.8. We recall the solution of the classical problem of worker–firm matching. Let $W = \{w_1, w_2, \dots\}$ be a set of workers and $F = \{f_1, f_2, \dots\}$ a set of firms. Each worker w_i has a list of firms (a subset of F) he/she would like to be hired by, with a strict preference ordering. Similarly, each firm has a list of workers (a subset of W) it would like to hire, with a strict preference ordering. Furthermore, a worker w_i may be hired by at most q_{w_i} firms (we allow several part time jobs), and each firm f_j has a quota q_{f_j} of maximum number of workers it can hire. By a contract we mean a worker–firm pair (w_i, f_j).

Now we define two choice maps C_W, C_F on $X := W \times F$ as follows. Applying the preceding example, for each worker w_i we construct a choice map C_{w_i} in $\{w_i\} \times F$, using the worker's preference ordering and the quota q_{w_i}, and for each firm f_j we construct a choice map C_{f_j} in $W \times \{f_j\}$, using the firm's preference ordering and the quota q_{f_j}. Finally, for any set $A \subset X := W \times F$ we define $C_W(A)$ and $C_F(A)$ by the formulas

$$C_W(A) := \cup_{w_i} C_{w_i}(A \cap \{w_i\} \times F)$$

and

$$C_F(A) := \cup_{f_j} C_{f_j}(A \cap W \times \{f_j\}).$$

All choice maps C_{w_i} and C_{f_j} satisfy the strengthened substitutes condition by Example 2.4. It follows easily (see [18, Proposition 3.12]) that both C_W and C_F are choice maps in $X = W \times F$ satisfying the strengthened substitutes condition. We may thus apply the above theorem to obtain a stable set.

3. Conditional matchings

As before, we denote by X the set of all possible contracts. By a *bloc* we simply mean a subset of X.

Definition 3.1. Given a family $\{X_j\}$ of blocs, a choice map $C : 2^X \to 2^X$ is called *admissible* if for each $A \subset X$ and for each bloc X_j we have either $X_j \subset C(A)$ or $X_j \cap C(A) = \varnothing$.

Lemma 3.2.

(i) *An admissible choice map remains admissible if we add one-point sets as blocks.*

(ii) *An admissible choice map remains admissible if we replace any two intersecting blocs by their union.*

Proof. Let C be an admissible choice map in X with blocks X_j.

(i) For any given $A \subset X$ and $x \in X$ we have either $x \in C(A)$ or $x \notin C(A)$, which may also be written in the form $\{x\} \subset C(A)$ or $\{x\} \cap C(A) = \varnothing$.

(ii) If two blocks X_j, X_k have a common element x, then for each $A \subset X$ we have either $X_j \cup X_k \subset C(A)$ or $(X_j \cup X_k) \cap C(A) = \varnothing$. Indeed, in case $x \in C(A)$ we have $X_j \subset C(A)$ and $X_k \subset C(A)$, while in case $x \notin C(A)$ we have $X_j \cap C(A) = \varnothing$ and $X_k \cap C(A) = \varnothing$ by definition of admissibility.

□

Assumption 3.3. In view of the preceding lemma we assume henceforth that the blocs X_j form a *partition* of X, i.e., they are disjoint and their union is X.

Under this assumption a choice function is admissible if and only if $C(A)$ is a union of blocs for each $A \subset X$.

In order to apply Theorem 2.5 for problems with blocs, we need to construct suitable admissible choice maps, satisfying the strengthened substitutes condition.

Given a choice map, the most natural way to construct an admissible choice map is the following: we accept the contracts of a bloc if and only if each contract of the bloc would individually be accepted in the absence of the bloc. This leads to the following

Definition 3.4. Given a choice map $C : 2^X \to 2^X$, we define an *induced* map $\overline{C} : 2^X \to 2^X$ by setting

$$\overline{C}(A) := \cup \{X_j \ : \ X_j \subset C(A)\}$$

for all $A \subset X$.

Proposition 3.5. *$\overline{C}$ is an admissible choice map.*

Proof. Since $\overline{C}(A) \subset C(A) \subset A$ for all A, $\overline{C}$ is a choice map. The admissibility follows by observing that each image $\overline{C}(A)$ is a union of blocks by definition. □

Remarks 3.6.

(i) If all blocs are one-point sets, then $\overline{C} = C$.

(ii) For any $A \subset X$ we have $\overline{C}(A) = \overline{C}(\overline{A})$ where $\overline{A}$ denotes the union of the blocs $X_j \subset A$. Hence we could simplify our map $\overline{C}$, without loss of information, by considering its restriction to the family of unions of blocs.

Unfortunately, the strengthened substitutes condition of a choice map may be lost when we use an induced choice map. We illustrate this by an example.

Example 3.7. We consider the worker–firm model with one firm and four workers, so that

$$X = \{(w_1, f_1), (w_2, f_1), (w_3, f_1), (w_4, f_1)\}.$$

For simplicity of notations we write w_i instead of (w_i, f_1), so that

$$X = \{w_1, w_2, w_3, w_4\}.$$

We assume that the firm may hire up to two workers and its preference ordering is $w_1 \succ w_2 \succ w_3 \succ w_4$. This gives rise to a choice map $C := C_F = C_{f_1} : 2^X \to 2^X$, satisfying the strengthened substitutes condition, obtained by the construction of Example 2.4 with the quota $q = 2$. We observe that

$$C(X) = \{w_1, w_2\} \quad \text{and} \quad C(\{w_1, w_3\}) = \{w_1, w_3\}.$$

Next we consider the induced admissible choice map $\overline{C} : 2^X \to 2^X$ corresponding to the two blocs $X_1 = \{w_1, w_3\}$ and $X_2 = \{w_2, w_4\}$. Consider the sets $A := X$ and $B := \{w_1, w_3\}$. Then $\overline{C}(A) = \varnothing$ because $C(A) = \{w_1, w_2\}$ contains none of the blocs $\{w_1, w_3\}$ and $\{w_2, w_4\}$, and $\overline{C}(B) = C(B) = \{w_1, w_3\}$ because $C(B) = \{w_1, w_3\}$ is a bloc.

Since

$$\overline{C}(A) = \varnothing \quad \text{and} \quad A \cap \overline{C}(B) = \{w_1, w_3\} \neq \varnothing,$$

we have

$$\overline{C}(A) \subset B \quad \text{but} \quad A \cap \overline{C}(B) \not\subset \overline{C}(A),$$

so that the induced admissible choice map does not satisfy the strengthened substitutes condition.

In view of this counterexample we choose another way to construct suitable choice maps by generalizing the construction of Example 2.4.

Definition 3.8. Let the blocs $X_1, \dots, X_n$ form a partition of X with the strict preference ordering $X_1 \succ X_2 \succ \cdots \succ X_n$, and fix a quota $q \geq 0$. For any given set $A \subset X$, we define recursively the sets $C_0(A), C_1(A), \dots, C_n(A)$ as follows.

First we set $C_0(A) := \varnothing$. Then, if $C_j(A)$ has already been defined for some $0 \leq j < n$, then we set

$$C_{j+1}(A) := \begin{cases} C_j(A) \cup X_{j+1} & \text{if } X_{j+1} \subset A \text{ and } |C_j(A) \cup X_{j+1}| \leq q, \\ C_j(A) & \text{otherwise.} \end{cases} \tag{3.1}$$

Finally, we define $C(A) := C_n(A)$. Observe that $C : 2^X \to 2^X$ is an admissible choice map by construction, and $|C(A)| \leq q$ for all $A \subset X$.

Unfortunately, this construction still does not always give choice maps satisfying the strengthened substitutes condition:

Example 3.9. As in Example 3.7, we consider again the worker–firm model with one firm and four workers, so that

$$X = \{w_1, w_2, w_3, w_4\},$$

and the firm may hire up to two workers. Now we assume that w_2 and w_3 form a bloc. Then the former ordering $w_1 \succ w_2 \succ w_3 \succ w_4$ gives rise naturally to the new ordering $\{w_1\} \succ \{w_2, w_3\} \succ \{w_4\}$ between the blocs. Applying the above construction with $q = 2$ we obtain an admissible choice map satisfying the equalities

$$C(\{w_2, w_3, w_4\}) = \{w_2, w_3\} \quad \text{and} \quad C(\{w_1, w_2, w_3, w_4\}) = \{w_1, w_4\}.$$

Hence for $A = \{w_2, w_3, w_4\}$ and $B = X$ we have

$$C(A) \subset B \quad \text{but} \quad A \cap C(B) = \{w_4\} \not\subset \{w_2, w_3\} = C(A).$$

There is, however, a practical sufficient condition ensuring the strengthened substitutes property:

Theorem 3.10. *Let the choice map be constructed as in (3.1). If the numbers of elements of X_j form a non-decreasing sequence, i.e.,*

$$|X_1| \leq |X_2| \leq \cdots \leq |X_n|, \tag{3.2}$$

then the choice map $C : 2^X \to 2^X$ satisfies the strengthened substitutes condition.

Remarks 3.11.

(i) If $X_1, \dots, X_n$ are all one-point sets, then condition (3.2) is obviously satisfied and our construction reduces to Example 2.4.

(ii) Condition (3.2) expresses a reasonable compromise in treating blocs: in order to ensure the existence of stable contract sets, priority is given to smaller blocs. This is also a kind of *anti-trust rule*. The intuition is that firms see groups of workers as more substitutes, the smaller their size. Indeed, smaller groups represent less constraints on the firms'preferences, hence increasing the overall assignment satisfaction.

Proof. Let $A, B \subset X$ be two sets satisfying $C(A) \subset B$; we have to show that $A \cap C(B) \subset C(A)$. Since both $C(A)$ and $C(B)$ are unions of some sets X_j by construction, this amounts to show that if $X_k \subset A \cap C(B)$ for some $1 \le k \le n$, then $X_k \subset C(A)$.

We use the notations of the second construction of C given above.

If $1 \le j < k$, then using (3.2) and the assumption $X_k \subset C(B)$ we get

$$|C_{j-1}(B)| + |X_j| \le |C_{k-1}(B)| + |X_k| = |C_k(B)| \le q.$$

If $X_j \subset B$, then we conclude that $X_j \subset C(B)$ by definition of $C_j(B)$. In particular, using our assumption $C(A) \subset B$ we obtain that

$$C_{k-1}(A) \subset C_{k-1}(B). \tag{3.3}$$

Since $X_k \subset A \cap C(B)$ by assumption, we have $X_k \subset A$, and

$$|C_{k-1}(B)| + |X_k| \le q$$

by definition of $C_k(B)$. Using (3.3) it follows that

$$|C_{k-1}(A)| + |X_k| \le q.$$

Since $X_k \subset A$, by definition of $C_k(A)$ we conclude that $X_k \subset C_k(A)$. □

Example 3.12. We return to the problem of blocs discussed in Example 3.7. We have $X = \{w_1, w_2, w_3, w_4\}$, $q = 2$ and the blocs $X_1 := \{w_1, w_3\}$ and $X_2 := \{w_2, w_4\}$. In view of the ordering $w_1 \succ w_2 \succ w_3 \succ w_4$ it is natural to order the two blocs by the relation $\{w_1, w_3\} \succ \{w_2, w_4\}$. This yields the choice map given by the formulas (see Remark 3.6 (ii))

$$C(\varnothing) = \varnothing, \quad C(\{w_1, w_3\}) = \{w_1, w_3\}\,, \quad C(\{w_2, w_4\}) = \{w_2, w_4\}$$

and

$$C(\{w_1, w_2, w_3, w_4\}) = \{w_1, w_3\}\,.$$

One may readily check that this choice map satisfies the strengthened substitutes condition. This also follows from Theorem 3.10 because condition (3.2) is obviously fulfilled here: the blocs have the same number of elements.

It is instructive to compare this choice map with the induced choice map in Example 3.7, which did not satisfy the strengthened substitutes condition:

$$\overline{C}(\varnothing) = \varnothing, \quad \overline{C}(\{w_1, w_3\}) = \{w_1, w_3\}\,, \quad \overline{C}(\{w_2, w_4\}) = \{w_2, w_4\}$$

and

$$\overline{C}(\{w_1, w_2, w_3, w_4\}) = \varnothing.$$

4. An adoption problem

We illustrate the usefulness of Theorem 3.10 by some model problems.

Problem 1. Consider the assignment of five children c_1, c_2, c_3, c_4, c_5 to five potential families f_1, f_2, f_3, f_4, f_5. Families f_1 and f_4 have the potential to adopt two children, families f_2, f_3 and f_5 can adopt only one.

The family preferences are as follows:

- Preference order of $f_1 : c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5$
- Preference order of $f_2 : c_3 \succ c_5 \succ c_2 \succ c_1 \succ c_4$
- Preference order of $f_3 : c_4 \succ c_2 \succ c_1 \succ c_5 \succ c_3$
- Preference order of $f_4 : c_4 \succ c_2 \succ c_5 \succ c_1 \succ c_3$
- Preference order of $f_5 : c_3 \succ c_4 \succ c_5 \succ c_1 \succ c_2$

Through their agencies, the ranking of children has been reported as follows:

- Preference order of $c_1 : f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5$
- Preference order of $c_2 : f_5 \succ f_1 \succ f_2 \succ f_3 \succ f_4$
- Preference order of $c_3 : f_5 \succ f_4 \succ f_3 \succ f_2 \succ f_1$
- Preference order of $c_4 : f_1 \succ f_5 \succ f_3 \succ f_4 \succ f_2$
- Preference order of $c_5 : f_1 \succ f_5 \succ f_4 \succ f_3 \succ f_2$

For the solution we set

$$C := \{c_1, c_2, c_3, c_4, c_5\}, \quad F := \{f_1, f_2, f_3, f_4, f_5\}$$

and we proceed in several steps.

Step 1. For each fixed child c_i we apply Example 2.4 to define a choice map C_{c_i} on $\{c_i\} \times F$ with Y given below. For brevity we write (i, j) instead of (c_i, f_j) in the preference relations.

- For child c_1 we choose $q = 1$,
$$Y := \{c_1\} \times \{f_1, f_2, f_3, f_4, f_5\},$$
$$(1,1) \succ (1,2) \succ (1,3) \succ (1,4) \succ (1,5).$$
- For child c_2 we choose $q = 1$,
$$Y := \{c_2\} \times \{f_1, f_2, f_3, f_4, f_5\},$$
$$(2,5) \succ (2,1) \succ (2,2) \succ (2,3) \succ (2,4).$$
- For child c_3 we choose $q = 1$,
$$Y := \{c_3\} \times \{f_1, f_2, f_3, f_4, f_5\},$$
$$(3,5) \succ (3,4) \succ (3,3) \succ (3,2) \succ (3,1).$$
- For child c_4 we choose $q = 1$,
$$Y := \{c_4\} \times \{f_1, f_2, f_3, f_4, f_5\},$$
$$(4,1) \succ (4,5) \succ (4,3) \succ (4,4) \succ (4,2).$$
- For child c_5 we choose $q = 1$,
$$Y := \{c_5\} \times \{f_1, f_2, f_3, f_4, f_5\},$$
$$(5,1) \succ (5,5) \succ (5,4) \succ (5,3) \succ (5,2).$$

Step 2. As in Example 2.8, we combine the three choice maps of the preceding step into a global choice map C_W on $W \times F$ by setting

$$C_W(A) := \bigcup_{i=1}^{5} C_{c_i}\left(A \cap (\{c_i\} \times F)\right)$$

for every $A \subset C \times F$.

Step 3. For each family f_j we apply Example 2.4 to define a choice map C_{f_j} on $C \times \{f_j\}$ with Ygiven below and still writing (i,j) instead of (c_i, f_j) for brevity.

- For family f_1 we choose $q = 2$,
$$Y := \{c_1, c_2, c_3, c_4, c_5\} \times \{f_1\},$$
$$(1,1) \succ (2,1) \succ (3,1) \succ (4,1) \succ (5,1).$$
- For family f_2 we choose $q = 1$,
$$Y := \{c_1, c_2, c_3, c_4, c_5\} \times \{f_2\},$$
$$(3,2) \succ (5,2) \succ (2,2) \succ (1,2) \succ (4,2).$$
- For family f_3 we choose $q = 1$,
$$Y := \{c_1, c_2, c_3, c_4, c_5\} \times \{f_3\},$$
$$(4,3) \succ (2,3) \succ (1,3) \succ (5,3) \succ (3,3).$$
- For family f_4 we choose $q = 2$,
$$Y := \{c_1, c_2, c_3, c_4, c_5\} \times \{f_4\},$$
$$(4,4) \succ (2,4) \succ (5,4) \succ (1,4) \succ (3,4).$$
- For family f_5 we choose $q = 1$,
$$Y := \{c_1, c_2, c_3, c_4, c_5\} \times \{f_5\},$$
$$(3,5) \succ (4,5) \succ (5,5) \succ (1,5) \succ (2,5).$$

Step 4. As in Example 2.8, we combine the three choice maps of the preceding step into a global choice map C_F on $W \times F$ by setting

$$C_F(A) := \bigcup_{j=1}^{5} C_{f_j}\left(A \cap (C \times \{f_j\})\right), \quad A \subset C \times F.$$

Step 5. The choice maps C_W and C_F satisfy the hypotheses of Theorem 2.5. Applying the algorithms described in [18, Remark 3.7] we obtain after some computation that the children-optimal and the family-optimal equilibria are the same:

- children c_1 and c_2 are adopted by family f_1;
- child c_3 is adopted by family f_5;
- child c_4 is adopted by family f_3;
- child c_5 is adopted by family f_4.

moveComputation of the children-optimal equilibrium. Starting with $X_0 := X$ we compute $Y_1, X_2, Y_3, X_4...$ by the formulae

$$Y_{n+1} := (X \setminus X_n) \cup C_W(X_n) \quad \text{and} \quad X_{n+1} := (X \setminus Y_n) \cup C_F(Y_n).$$

The results are summarized in the following table:

c_i	f_j	X_0	Y_1	X_2	Y_3	X_4	Y_5	X_6	Y_7	X_8	S
1	1	x	x	x	x	x	x	x	x	x	x
1	2	x		x		x		x		x	
1	3	x		x		x		x		x	
1	4	x		x		x		x		x	
1	5	x		x		x		x		x	
2	1	x		x	x	x	x	x	x	x	x
2	2	x		x		x		x		x	
2	3	x		x		x		x		x	
2	4	x		x		x		x		x	
2	5	x	x		x		x		x		
3	1	x		x		x		x		x	
3	2	x		x		x		x		x	
3	3	x		x		x		x		x	
3	4	x		x		x		x		x	
3	5	x	x	x	x	x	x	x	x	x	x
4	1	x	x	x	x		x		x		
4	2	x		x		x		x		x	
4	3	x		x		x		x	x	x	x
4	4	x		x		x		x		x	
4	5	x		x		x	x		x		
5	1	x	x		x		x		x		
5	2	x		x		x		x		x	
5	3	x		x		x		x		x	
5	4	x		x		x	x	x	x	x	x
5	5	x		x	x		x		x		

For example, the table shows that

$$Y_1 = \{(1,1),(2,5),(3,5),(4,1),(5,1)\};$$

the interpretation of the other columns here and in the subsequent tables is similar.

The children-optimal equilibrium is thus the following:

- children c_1 and c_2 are adopted by family f_1;
- child c_3 is adopted by family f_5;
- child c_4 is adopted by family f_3;
- child c_5 is adopted by family f_4.

Next we compute the family-optimal solution:

C_i	f_j	Y_0	X_1	Y_2	X_3	Y_4	X_5	Y_6	X_7	Y_8	X_9	Y_{10}	X_{11}	Y_{12}	S
1	1	x	x	x	x	x	x	x	x	x	x	x	x	x	x
1	2	x		x		x		x	x		x		x		
1	3	x		x		x		x		x		x		x	
1	4	x		x	x		x		x		x		x		
1	5	x		x		x		x		x		x		x	
2	1	x	x	x	x	x	x	x	x	x	x	x	x	x	x
2	2	x		x		x	x		x		x		x		
2	3	x		x		x		x		x		x		x	
2	4	x	x		x		x		x		x		x		
2	5	x		x		x		x		x		x		x	
3	1	x		x		x		x		x		x		x	
3	2	x	x		x		x		x		x		x		
3	3	x		x		x		x		x		x		x	
3	4	x		x		x	x		x		x		x		
3	5	x	x	x	x	x	x	x	x	x	x	x	x	x	x
4	1	x		x		x		x		x		x		x	
4	2	x		x		x		x		x	x		x		
4	3	x	x	x	x	x	x	x	x	x	x	x	x	x	x
4	4	x	x		x		x		x		x		x		
4	5	x		x		x		x		x		x		x	
5	1	x		x		x		x		x		x		x	
5	2	x		x	x		x		x		x		x		
5	3	x		x		x		x		x		x		x	
5	4	x		x	x	x	x	x	x	x	x	x	x	x	x
5	5	x		x		x		x		x		x		x	

The family-optimal equilibrium is thus the same as above.

Problem 2. We reconsider the previous problem by adding some conditions. We assume that c_1, c_2, c_3 are brothers and sisters, while c_4 and c_5 are twins. Children are concerned about separation but agencies have some information about families location and try to allocate children according to their wish to the best of their ability.

We add the following condition: the twins c_4 and c_5 have to be adopted together. This implies that they may be adopted only by families f_1 or f_4.

We modify the preference relations of the previous subsection as follows: we remove children c_4 and c_5 from the preference orders of families f_2, f_3, f_5, and we replace the preference orders c_4 and c_5 by the preference order $f_1 \succ f_4$ for the couple $\{c_4, c_5\}$. Thus now we have the following relations:

- Preference order of $f_1 : c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5$
- Preference order of $f_2 : c_3 \succ c_2 \succ c_1$
- Preference order of $f_3 : c_2 \succ c_1 \succ c_3$
- Preference order of $f_4 : c_4 \succ c_2 \succ c_5 \succ c_1 \succ c_3$
- Preference order of $f_5 : c_3 \succ c_1 \succ c_2$

- Preference order of $c_1 : f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5$
- Preference order of $c_2 : f_5 \succ f_1 \succ f_2 \succ f_3 \succ f_4$
- Preference order of $c_3 : f_5 \succ f_4 \succ f_3 \succ f_2 \succ f_1$
- Preference order of $\{c_4, c_5\} : f_1 \succ f_4$

While the first problem could be solved by the usual algorithm as in [18], for the solution of Problem 2 we also need Theorem 3.10 above. Using the modified preference relations, keeping the same quotas as in Problem 1, and taking quota $q = 1$ for $\{c_4, c_5\}$, we construct the corresponding choice maps as before. Thanks to Theorem 3.10 we may apply the usual algorithms. It turns out that the children-optimal and the family-optimal equilibria coincide again, but they are different from that of Problem 1:

- children c_1 and c_2 are adopted by family f_1;
- child c_3 is adopted by family f_5;
- children c_4 and c_5 are adopted by family f_4.

Problem 3. We consider Problem 2 with a less restrictive condition: families f_1 or f_4 may only adopt twins c_4 and c_5 together. This does not rule out the possibility that the twins are adopted separately by the other three families f_2, f_3 or f_5.

With respect to Problem 1 we modify the preference orderings of f_1 and f_4 as follows:

- Preference order of $f_1 : c_1 \succ c_2 \succ c_3 \succ \{c_4, c_5\}$
- Preference order of $f_4 : c_2 \succ c_1 \succ c_3 \succ \{c_4, c_5\}$

In order to apply Theorem 3.10 we have to give the couple $\{4, 5\}$ the least preference: otherwise condition (3.2) would not be satisfied.

Using the modified preference relations and keeping the same quotas, our algorithms show that the children-optimal and the family-optimal equilibria coincide again:

- children c_1 and c_2 are adopted by family f_1;
- child c_3 is adopted by family f_5;
- child c_4 is adopted by family f_3;
- child c_5 is adopted by family f_2.

The solution is thus different from the solutions of preceding two problems.

We summarize our results in the following table:

Child	Problem 1	Problem 2	Problem 3
c_1	f_1	f_1	f_1
c_2	f_1	f_1	f_1
c_3	f_5	f_5	f_5
c_4	f_3	f_4	f_3
c_5	f_4	f_4	f_2

As we have already observed, the children-optimal and the family-optimal solutions will coincide in all three problems. Furthermore, children c_1, c_2 and c_3 get the same adoptive parents in all cases. On the other hand, the results for children c_4 and c_5 are different in each case.

move

5. Extension: Schedule matchings

We conclude our paper by considering the more complex problem of *schedule matchings*. We illustrate the problem by an example.

Example 5.1. We return to the worker-firm model. We assume that the hiring is made for each day d_k of a set D of days, e.g., of a week or a month. The contracts are the elements of the set $X := W \times F \times D$. Each worker w_i has a set of contracts belonging to $\{w_i\} \times F \times D$ with a quota q_{w_i} of maximum number of contracts and a strict preference ordering on the subset. Similarly, each firm f_j has a set of contracts belonging to $W \times \{f_j\} \times D$ with a quota q_{f_j} of maximum number of contracts and a strict preference ordering on the subset. Finally, we may have a global quota q of the maximum number of all accepted contracts.

For problems of this type a general and efficient algorithm has been developed in [18]. The question is whether this algorithm can be extended in order to handle blocs, too.

The mathematical framework is the following. As before, let $\{X_1, \dots, X_n\}$ be a finite family of disjoint subsets of X, $X_1 \succ X_2 \succ \cdots \succ X_n$ a complete ordering of its elements, and q a nonnegative integer (*quota*).

The novelty is that we also have another finite family $\{Y_1, \dots, Y_m\}$ of disjoint subsets of X, and corresponding nonnegative integers $q_1, \dots, q_m$ (*local quotas*). (For instance, in the case of Example 5.1 Y_k may be the set of contracts on day d_k.)

For any given set $A \subset X$, we define recursively the sets $C_0(A), C_1(A), \dots, C_n(A)$ as follows.

First we set $C_0(A) := \varnothing$. Then, if $C_i(A)$ has already been defined for some $0 \le i < n$, then we set

$$C_{i+1}(A) := \begin{cases} C_i(A) \cup X_{i+1} & \text{if } X_{i+1} \subset A,\ |C_i(A) \cup X_{i+1}| \le q \\ & \text{and } |(C_i(A) \cup X_{i+1}) \cap Y_j| \le q_j \text{ for all } j, \\ C_i(A) & \text{otherwise.} \end{cases}$$

Finally, we define $C(A) := C_n(A)$. We observe that $C(A) \subset A$ for all A, so that $C : 2^X \to 2^X$ is a choice map. We observe also that $|C(A)| \le q$ and $|C(A) \cap Y_j| \le q_j$ for all j.

One may wonder whether condition (3.2) of Theorem 3.10 ensures the strengthened substitutes property of this more general choice map, too. A moment of reflection shows that we should also assume that none of the blocs is splitted by the sets Y_j, i.e.,

$$\text{either} \quad X_i \cap Y_j = \varnothing \quad \text{or} \quad X_i \subset Y_j$$

for all i, j. However, the following example shows that these two conditions are still not sufficient.

Example 5.2. We set $X = X_1 \cup X_2 \cup X_3 = Y_1 \cup Y_2$ with

$$\begin{aligned} &X = \{a, b, c, d, e, f\}, \\ &X_1 = \{a\}, \quad X_2 = \{b, c\}, \quad X_3 = \{d, e, f\}, \\ &Y_1 = X_1 \cup X_2 = \{a, b, c\}, \quad Y_2 = X_3 = \{d, e, f\}, \end{aligned}$$

the preference ordering $X_1 \succ X_2 \succ X_3$ and quotas $q_1 = 2$, $q_2 = 3$, $q = 4$.

Then for the sets $A := \{b, c, d, e, f\}$ and $B := X = \{a, b, c, d, e, f\}$ we have $C(A) = \{b, c\}$ and $C(B) = \{a, d, e, f\}$. In particular,

$$C(A) \subset B \quad \text{but} \quad A \cap C(B) = \{d, e, f\} \not\subset \{b, c\} = C(A),$$

so that the strengthened substitutes condition is not satisfied.

The generalization of Theorem 3.10 to schedule matchings seems to be a challenging open problem.

Acknowledgement. The authors thank the referee for his valuable comments and suggestions to improve the presentation of our results.

References

[1] A. Alkan, D. Gale, *Stable schedule matching under revealed preference*, J. Economic Theory 112 (2003), 289–306.
[2] O. Aygün, T. Sönmez, *The importance of irrelevance of rejected contracts in matching under weakened substitutes conditions,* Mimeo, Boston College, 2012.
[3] O. Aygün, T. Sönmez, *Matching with contracts: The critical role of irrelevance of rejected contracts,* Mimeo, Boston College, 2012.
[4] D. Dimitrov, E. Lazarova, *Two-sided coalitional matchings*, Mathematical Social Sciences, 62 (2011), 1, 46–54.
[5] F. Echenique, J. Ovideo, *Core many-to-one matchings by fixed-point methods*, Journal of Economic Theory 115 (2004), 358–376.
[6] F. Echenique, J. Ovideo, *A theory of stability in many-to-many matching markets*, Theoretical Economics 1 (2006), 233–273.
[7] F. Echenique, M. B. Yenmez, *A solution to matching with preferences over colleagues*, Games Econom. Behav. 59 (2007), no. 1, 46–71.
[8] T. Fleiner, *A fixed-point approach to stable matchings and some applications*, Math. Oper. Res. 28 (2003), 103–126.
[9] T. Fleiner, *Stable matchings through fixed points and graphs*, Ann. Univ. Sci. Budapest. Sect. Math. 51 (2008), 69–116 (2009).
[10] D. Gale, L. S. Shapley, *College admissions and the stability of marriage*, Amer. Math. Monthly 69 (1962), 1, 9–15.
[11] C.-J. Haake, B. Klaus, *Stability and Nash implementation in matching markets with couples*, Theory and Decision 69 (2010), no. 4, 537–554,
[12] J. W. Hatfield, F. Kojima, *Matching with contracts: comment*, Amer. Economic Review 98 (2008), 1189–1194.
[13] J. W. Hatfield, P. Milgrom, *Matching with contracts*, Amer. Economic Review 95 (2005), 913–935.
[14] A. S. Kelso jun., V. P. Crawford, *Job matching, coalition formation, and gross substitutes* , Econometrica 50 (1982), 1483–1504.
[15] B. Klaus, F. Klijn, *Stable matchings and preferences of couples*, J. Economic Theory 121 (2005), 1 , 75–106.
[16] B. Klaus, F. Klijn, *Paths to stability for matching markets with couples*, Games Econom. Behav. 58 (2007), no. 1, 154–171,
[17] B. Klaus, F. Klijn, *Fair and efficient student placement with couples*, Internat. J. Game Theory 36 (2007), no. 2, 177–207.
[18] V. Komornik, Zs. Komornik, C. Viauroux, *Stable schedule matchings*, Acta Math. Hungar. 135 (2012), 1–2, 67–79.
[19] M. Pycia, *Stability and preference alignment in matching and coalition formation*, Econometrica 80 (2012), 1, 323–362.
[20] A. E. Roth, *Stability and polarization of interests in job matching*, Econometrica 52 (1984), 1, 47–57.
[21] A. E. Roth, *The evolution of the labor market for medical interns and residents*, J. Polit. Econ. 92 (1984), 991–1016.
[22] A. E. Roth, *Conflict and coincidence of interest in job matching: some new results and open questions*, Math. Oper. Res. 10 (1985), 3, 379–389.
[23] A. E. Roth, *The college admissions problem is not equivalent to the marriage problem*, J. Economic Theory 36 (1985), 277–288.
[24] A. E. Roth, M. Sotomayor, *Two-sided matching. A study in game-theoretic modeling and analysis*, Econometric Society Monographs, 18. Cambridge University Press, Cambridge, 1990.

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