STABLE SCHEDULE MATCHINGS

VILMOS KOMORNIK, ZSOLT KOMORNIK, AND CHRISTELLE K. VIAUROUX

ABSTRACT. In order to treat a natural schedule matching problem related with worker-firm matchings, we generalize some theorems of Baiou–Balinski and Alkan–Gale by applying a fixed point method of Fleiner.

1. INTRODUCTION

Since the pioneering paper of Gale and Shapley [12] on *stable matchings,* many studies have been devoted to the adaptations and the generalizations of their algorithm. They have found use in diverse economic applications ranging from labor markets to college admissions or even kidney exchanges.

In these two-sided matching markets, two sets of agents have preferences over the opposite set: on one side of the market, there are individuals (students, interns or employees) and on the other side there are institutions (colleges, hospitals or firms). A "stable match" is realized when all agents have been matched with the opposite side such that neither could obtain a more mutually beneficial match on their own.

The original strict preference ordering assumptions proved to be too restrictive for many real world problems. Following an influential contribution of Kelso and Crawford [5], Roth [16] and Blair [4] made a systematic study of a more flexible approach based on *choice functions.* The monograph of Roth and Sotomayor [17] provides an overview of the state of the art up to 1990 and it still serves as an excellent introduction to the subject. Feder [8], Subramanian [18] and Adachi [1] discovered a close relationship between stable matchings and fixed points of set-valued maps. Fleiner [11], [10] demonstrated that many classical results may be obtained by a straightforward application of an old theorem of Knaster [15] and Tarski [19], [20].

In their paper, Kelso and Crawford introduced an important *substitutability property.* This, together with a special preference relation yields a choice map for which the theory applies. Many papers have followed by adapting the theory to more realistic and more complex problems; we cite Hatfield and Milgrom [13], Echenique and Ovideo [6], [7], Klaus and Walzl [14] among the most recent ones: many others figure in their references.

Baiou and Balinski [3] introduced the notion of *schedule matching* which made it possible to consider, as a part of the contract not only the hiring of a particular worker by a particular firm, but also the number of hours of employment of the worker in the firm. In their setting, "stability" means that no pair of opposite agents can increase their hours together either due to unused capacity or by giving up hours with less desirable partners. They assumed that all agents have strict

2000 *Mathematics Subject Classification.* Primary: 91B68; Secondary: 90C27.
Key words and phrases. stable matching; choice function; schedule matching.

preference orderings. Alkan and Gale [2] extended their model by using incomplete revealed preference ordering via choice functions instead.

In this paper, we generalize the notion of schedule matching of Baiou and Balinski [3] to allow for schedule and preference constraints on each side of the market. We define a choice map of a new kind for each agent on the acceptable opposite side agent(s), possible days and (combinations of) restrictions or "subsets" placed on the opposite side agent and/or days worked. In particular, our framework allows for possible quotas placed by workers on firms and days worked, allowing him to work part-time for different firms on the same day or on different days, excluding some firms on some given days or excluding some days of work. In the same manner, it allows firms to adjust their labor force on certain days depending on their anticipated activity, or on the requirements associated to different activities on different days (or the same day). We show that the allocation of days, firms and workers is stable in the sense that given their schedule constraints, their preference orderings and constraints, there is no better schedule for both parties; moreover the stable allocation is shown to be worker optimal or firm optimal. This is done by a new, general construction of choice maps satisfying a different kind of strengthened substitutes condition, for which a classical fixed point theorem may be applied. We provide the algorithm that can be used to obtain the optimal allocation: we will solve a deliberately complex example to explain its technical execution.

The structure of the paper is the following. In Section 2 we formulate a model problem which will motivate our research and which may have natural real-word applications. In Section 3 we present the mathematical framework for our model. In Section 4 we solve our model problem and we also explain how our results cover some of the theorems of Alkan and Gale [2].

2. Schedule matching problems

In order to illustrate the novelty of the present work we begin by recalling the first example of Gale and Shapley [12]. They considered three women: w_1, w_2, w_3 and three men: f_1, f_2, f_3 with the following preference orders (for brevity we set $f_{i+3} := f_i$ and $w_{i+3} := w_i$ for all i):

- Preference order of w_i: $f_i \succ f_{i+1} \succ f_{i+2}$, $\quad i = 1, 2, 3$;
- Preference order of f_i: $w_{i+1} \succ w_{i+2} \succ w_{i+3}$, $\quad i = 1, 2, 3$.

They looked for the possibilities of marrying all six people in a stable way. Instability would occur if there were a woman and a man, not married to each other who would prefer each other to their actual mates. It turns out that there are three solutions:

- each woman gets her first choice: (w_1, f_1), (w_2, f_2), (w_3, f_3);
- each man gets his first choice: (w_1, f_3), (w_2, f_1), (w_3, f_2);
- everyone get her or his second choice: (w_1, f_2), (w_2, f_3), (w_3, f_1).

Now let us modify the problem to a simple job market problem as follows. Consider three workers: w_1, w_2, w_3 and three firms: f_1, f_2, f_3 with the same preference orders for hiring as above. Furthermore, assume that hiring is for two different days: d_1, d_2, with the following additional preferences and requirements:

- for each worker–firm pair (w_i, f_j), the worker prefers d_1 to d_2 and the firm prefers d_2 to d_1;

- each worker may be hired by at most one firm on each given day (maybe different firms on different days);
- each firm may hire at most two workers per day; if they hire two workers for one given day, then they cannot hire anybody for the other day;
- no firm may hire the same worker for both days.

We are looking for a stable set of contracts, i.e., for a set S of triplets (w_i, f_j, d_k) having the following properties:

- each contract $(w_i, f_j, d_k) \in S$ is acceptable for both w_i and f_j;
- for any other contract $(w_i, f_j, d_k) \notin S$, w_i or f_j (or both) prefers her/his contracts in S to this new one.

In the following section we recall some basic results from the theory of matchings and we construct a choice map of a new kind, well adapted to the kind of problems we aim to solve. The solution of our model problem is then given in Section 4. The method of our paper may be applied for much more complex problems as well.

3. Existence of stable schedule matchings

In this section we recall some basic notions from the theory of matchings.

Given a set X, we denote by 2^X the set of all subsets of X. By a *choice map* in X we mean a function $C : 2^X \to 2^X$ satisfying

$$C(A) \subset A \quad \text{for all} \quad A \subset X. \tag{3.1}$$

In economic applications X is the set of all possible contracts, and for a given set A of proposed contracts, $C(A)$ denotes the set of accepted contracts by some given rules of the market.

In order to ensure the existence of stable sets of contracts we impose the following assumption on the choice maps:

Definition 3.1. A choice map $C : 2^X \to 2^X$ satisfies the *strengthened substitutes condition* if

$$A, B \subset X \quad \text{and} \quad C(A) \subset B \Longrightarrow A \cap C(B) \subset C(A). \tag{3.2}$$

This means that if a contract is rejected from some proposed set A of contracts, then it will also be rejected from every other proposed set B which contains the accepted contracts.

Our notion differs from the strong substitutes condition of Echenique and Ovideo [6], [7] and of Klaus and Walzl [14] because our definition is purely set-theoretical: we do not use any preference relations. In fact, we could introduce preference relations leading to our choice map. But it would be artificial because already for a small size problem we have a very large number of suitable preference relations, and they would only hide the essential features of the problem.

The following proposition clarifies this notion:

Proposition 3.2. *A choice map $C : 2^X \to 2^X$ satisfies the strengthened substitutes condition if and only if it is* consistent *or satisfies the* path independence property*:*

$$C(A) \subset B \subset A \Longrightarrow C(B) = C(A) \tag{3.3}$$

and satisfies the substitutes condition*:*

$$A \subset B \Longrightarrow A \cap C(B) \subset C(A). \tag{3.4}$$

Proof. Let $C : 2^X \to 2^X$ satisfy the strengthened substitutes condition. Then it satisfies (3.4) because $C(A) \subset A$ for every choice map.

In order to prove (3.3) first we infer from our hypothesis $C(A) \subset B \subset A$ and from the choice map property $C(B) \subset B$ that $C(B) \subset A$. Therefore, applying (3.2) and using both inclusions $C(A) \subset B$ and $C(B) \subset A$ we have

$$C(A) \subset B \Longrightarrow A \cap C(B) \subset C(A) \Longrightarrow C(B) \subset C(A)$$

and

$$C(B) \subset A \Longrightarrow B \cap C(A) \subset C(B) \Longrightarrow C(A) \subset C(B),$$

so that $C(A) = C(B)$.

Now assume that $C : 2^X \to 2^X$ satisfies (3.3) and (3.4), and consider two sets satisfying $C(A) \subset B$. We have to prove that $A \cap C(B) \subset C(A)$.

Since $A \subset A \cup B$, applying (3.4) we obtain that

$$A \cap C(A \cup B) \subset C(A). \tag{3.5}$$

The proof will be completed by showing that $C(A \cup B) = C(B)$.

Using the hypothesis $C(A) \subset B$ we deduce from (3.5) that $A \cap C(A \cup B) \subset B$ and hence that $C(A \cup B) \subset B$. Therefore $C(A \cup B) \subset B \subset A \cup B$; applying (3.3) we conclude that $C(A \cup B) = C(B)$. □

Example 3.3.

(a) For any fixed set $Y \subset X$ the formula $C(A) := A \cap Y$ defines a choice map on X, satisfying (3.2). This example illustrates a situation where some contracts are unacceptable to certain agents.

(b) More generally, given a finite subset $Y \subset X$, a nonnegative integer q (called *quota*) and a strict preference ordering $y_1 \succ y_2 \succ \cdots$ on Y, we define a map $C(A)$ for any given $A \subset X$ as follows. If Card $(A \cap Y) \le q$ (the symbol Card stands for the number of elements), then we set $C(A) := A \cap Y$. If Card $(A \cap Y) > q$, then let $C(A)$ be the set of the first q elements of $A \cap Y$ according to the ordering of Y. Then $C : 2^X \to 2^X$ is a choice map on X, satisfying (3.2).

Choice maps of this kind are frequently used in classical matching problems such as the marriage problem, the college admission problem and various many-to-many matching problems; see, e.g., [2], [17] and the references of the latter.

Now assume that there are two competing sides, for example *workers and firms* and correspondingly two choice functions $C_W, C_F : 2^X \to 2^X$. We adopt here the following equilibrium concept:

Definition 3.4. A set S of contracts is said to be *stable* if there exist two sets $S_W, S_F \subset X$ satisfying the following three conditions:

$$S_W \cup S_F = X; \tag{3.6}$$

$$C_W(A) = S \quad \text{for every} \quad S \subset A \subset S_W; \tag{3.7}$$

$$C_F(A) = S \quad \text{for every} \quad S \subset A \subset S_F. \tag{3.8}$$

Stable contract sets represent acceptable compromises.

The following proposition shows that this definition is equivalent to a usual one:

Proposition 3.5. *Let $C_W, C_F : 2^X \to 2^X$ be two choice maps satisfying the strengthened substitutes condition. Then a set S is stable if and only if it is* individually rational:

$$C_W(S) = S = C_F(S), \tag{3.9}$$

and it is not blocked by any other contract*, i.e., for each $x \in X$ we have*

$$\text{either} \quad C_W(S \cup \{x\}) = S \quad \text{or} \quad C_F(S \cup \{x\}) = S \quad \text{(or both)}. \tag{3.10}$$

Proof. If S is a stable set, then (3.9)–(3.10) follow from (3.6)–(3.8). Now assume (3.9) and (3.10). Setting

$$S_W := \{x \in X \ : \ C_W(S \cup \{x\}) = S\} \quad \text{and} \quad S_F := \{x \in X \ : \ C_F(S \cup \{x\}) = S\}$$

we have $S \subset S_W$ and $S \subset S_F$ by (3.9), and $S_W \cup S_F = X$ by (3.10). Since C_W and C_F are consistent by Proposition 3.2, it remains to show that $C_W(S_W) = S = C_F(S_F)$.

If $x \in S_W \setminus S$, then applying (3.2) and using (3.10) we deduce from the inclusion $C_W(S \cup \{x\}) \subset S_W$ that

$$(S \cup \{x\}) \cap C_W(S_W) \subset C_W(S \cup \{x\}) = S$$

and hence $x \notin C_W(S_W)$. We have thus $C_W(S_W) \subset S$. Applying (3.2) again we deduce from this last inclusion that

$$S_W \cap C_W(S) \subset C_W(S_W).$$

Since $C_W(S) = S$ by (3.9) and since we already know that $S \subset S_W$, it follows that $S \subset C_W(S_W)$, so that finally $C_W(S_W) = S$. The proof of $C_F(S_F) = S$ is similar. □

In view of Propositions 3.2 and 3.5 the following theorem is equivalent to a theorem of Fleiner [11], [10]:

Theorem 3.6. *If the choice maps $C_W, C_F : 2^X \to 2^X$ satisfy the strengthened substitutes condition, then there exists at least one stable set of contracts.*

Remark 3.7. We recall from [11], [10] that if X is a finite set, then the proof of the theorem provides an efficient algorithm to find a stable set. Starting with $X_0 := X$ we compute successively $Y_1, X_2, Y_3, X_4, \dots$ by the formulas

$$Y_{n+1} := (X \setminus X_n) \cup C_W(X_n) \quad \text{and} \quad X_{n+1} := (X \setminus Y_n) \cup C_F(Y_n).$$

There exists a first index $n \geq 1$ such that $X_{n-1} = X_{n+1}$, and then $S := C_W(X_{n-1})$ is the worker-optimal stable set.

Similarly, starting with $Y_0 := X$ we may compute successively X_1, Y_2, X_3, $Y_4, \dots$ by the same recursive formulas. There exists a first index $n \geq 1$ such that $Y_{n-1} = Y_{n+1}$, and then $S := C_F(Y_{n-1})$ is the firm-optimal stable set.

In order to apply Theorem 3.6 for the solution of the problem stated in Section 2, we generalize the choice map construction of Example 3.3.

Let we are given a finite subset $Y \subset X$, a family $\{Y_n\}$ of subsets $Y_n \subset X$, and corresponding nonnegative integers (called *quotas*) q and q_n. We assume that the sets $Y_n \cap Y$ are disjoint. Furthermore, let be given a strict preference ordering $y_1 \succ y_2 \succ \cdots$ on Y. Given any set $A \subset X$, we define a non-decreasing sequence $C_0(A) \subset C_1(A) \subset \cdots$ of subsets of $A \cap Y$ by recursion as follows. First we set

$C_0(A) = \varnothing$. If $C_{k-1}(A)$ has already been defined for some k, then we set $C_k(A) := C_{k-1}(A) \cup \{y_k\}$ if

$$y_k \in A,$$
$$\text{Card } C_{k-1}(A) < q,$$
$$\text{Card } (C_{k-1}(A) \cap Y_n) < q_n \text{ if } y_k \in Y_n;$$

otherwise we set $C_k(A) := C_{k-1}(A)$. Finally, we define $C(A) := \cup C_k(A)$.

Remark 3.8. It follows from the construction that

$$C(A) \subset A \cap Y; \tag{3.11}$$
$$\text{Card } C(A) \le q; \tag{3.12}$$
$$\text{Card } (C(A) \cap Y_n) \le q_n \text{ for all } n. \tag{3.13}$$

We have the following result:

Theorem 3.9. *$C : 2^X \to 2^X$ is a choice map satisfying the strengthened substitutes condition.*

Remark 3.10.

(a) If $q_n \ge q$ or if $q_n \ge \text{Card } Y_n$ for some n, then we may eliminate Y_n and q_n without changing the construction.
(b) If there are no sets Y_n, then our construction reduces to Example 3.3 (b).
(c) If, moreover, $q \ge \text{Card } Y$, then our construction reduces to Example 3.3 (a). (In this case the choice of the order relation is irrelevant.)
(d) Instead of a finite subset $Y \subset X$, we can also consider arbitrary subsets $Y \subset X$ with a well-ordered preference relation: the construction and the proof of the theorem remain valid.
(e) In the absence of the sets Y_n and the quotas q_n our construction is a special case of the classical matroid greedy algorithm [9]. The latter may be probably generalized so as to include our construction in full generality.

Example 3.11. The disjointness condition is necessary. To show this, consider the sets $X = Y = \{a, b, c\}$, $Y_1 = \{a, b\}$, $Y_2 = \{b, c\}$ with the quotas $q = 2$, $q_1 = q_2 = 1$ and the preference order $a \succ b \succ c$. Then for $A = \{b, c\}$ and $B = \{a, b, c\}$ we have $C(A) = \{b\}$ and $C(B) = \{a, c\}$, so that $A \subset B$ but $A \cap C(B) \not\subset C(A)$.

Proof of Theorem 3.9. The choice map C remains the same if we change each Y_n to $Y_n \cap Y$ in the construction. The choice map does not change either if we complete the family $\{Y_n\}$ with $Y' := Y \setminus \cup Y_n$ corresponding to the quota $q' := \text{Card } Y'$. Without loss of generality we assume henceforth that $\{Y_n\}$ is a *partition* of Y, i.e., Y is the *disjoint union* of the sets Y_n.

Let $A, B \subset X$ be two sets satisfying $C(A) \subset B$; we have to show that if $y_k \in A \cap C(B)$ for some k, then $y_k \in C(A)$.

First we establish by induction on j the following inequalities:

$$\text{Card } (C_j(A) \cap Y_n) \le \text{Card } (C_j(B) \cap Y_n) \text{ for all } n, \quad j = 0, \dots, k-1. \tag{3.14}$$

For $j = 0$ our claim reduces to the trivial equality $0 = 0$. Assuming that the inequalities hold until some $j < k - 1$, consider the (unique) index m for which $y_{j+1} \in Y_m$. For each $n \ne m$ we have

$$C_j(A) \cap Y_n = C_{j+1}(A) \cap Y_n \text{ and } C_j(B) \cap Y_n = C_{j+1}(B) \cap Y_n$$

and therefore

$$\text{Card }(C_{j+1}(A) \cap Y_n) \leq \text{Card }(C_{j+1}(B) \cap Y_n)$$

by our induction hypothesis. For $n = m$ the only critical case is when

$$y_{j+1} \in C_{j+1}(A) \setminus C_{j+1}(B).$$

Since $y_{j+1} \in C(A)$ implies $y_{j+1} \in B$ and since

$$\text{Card } C_j(B) \leq \text{Card } C_{k-1}(B) = \text{Card } C_k(B) - 1 \leq q - 1$$

because $y_k \in C(B)$, by the construction this can only happen if

$$\text{Card }(C_j(A) \cap Y_m) \leq q_m - 1 \text{ and Card }(C_j(B) \cap Y_m) = q_m.$$

But then we have

$$\begin{aligned}
\text{Card }(C_{j+1}(A) \cap Y_m) &= \text{Card }(C_j(A) \cap Y_m) + 1 \\
&\leq q_m \\
&= \text{Card }(C_j(B) \cap Y_m) \\
&= \text{Card }(C_{j+1}(B) \cap Y_m)
\end{aligned}$$

as required. This completes the proof of the relations (3.14).

Since $y_k \in A \cap C(B)$, we have $y_k \in A$. Furthermore, since $C(A) \subset Y$ and the sets Y_n form a partition of Y, it follows from (3.14) that

$$\begin{aligned}
\text{Card } C_{k-1}(A) &= \cup_n \text{Card }(C_{k-1}(A) \cap Y_n) \\
&\leq \cup_n \text{Card }(C_{k-1}(B) \cap Y_n) \\
&= \text{Card } C_{k-1}(B) \\
&= \text{Card } C_k(B) - 1 \\
&\leq q - 1.
\end{aligned}$$

Furthermore, in case $y_k \in Y_n$ we have

$$(C_k(B) \cap Y_n) \setminus (C_{k-1}(B) \cap Y_n) = \{y_k\}$$

and therefore

$$\begin{aligned}
\text{Card }\ (C_{k-1}(A) \cap Y_n) &\leq \text{Card }\ (C_{k-1}(B) \cap Y_n) \\
&= \text{Card }\ (C_k(B) \cap Y_n) - 1 \\
&\leq q_n - 1.
\end{aligned}$$

Summarizing, the conditions (3.11)–(3.13) are satisfied and we conclude that $y_k \in C(A)$ by construction. This completes the proof. □

The following result enables us to combine individual choice functions satisfying the strengthened substitutability condition into a *global* choice function having the same property.

Proposition 3.12. *Given a partition $X = \cup_i X_i$ of a set X and choice functions $C_i : 2^{X_i} \to 2^{X_i}$ for each i, we define a choice function $C : 2^X \to 2^X$ by the formula*

$$C(A) := \cup_i C_i(A \cap X_i).$$

Then C satisfies the strengthened substitutability condition if and only if each C_i satisfies the strengthened substitutability condition.

Proof. If $C(A) \subset B$, then setting $A_i := A \cap X_i$ and $B_i := B \cap X_i$ we have

$$\begin{aligned} A \cap C(B) \subset C(A) &\Longleftrightarrow (A \cap C(B)) \cap X_i \subset C(A) \cap X_i \text{ for each } i \\ &\Longleftrightarrow A_i \cap C_i(B_i) \subset C_i(A_i) \text{ for each } i. \end{aligned}$$

□

4. Solution of the model problem

We set

$$W := \{w_1, w_2, w_3\}, \quad F := \{f_1, f_2, f_3\}, \quad D := \{d_1, d_2\},$$

and we proceed in several steps.

Step 1. For each fixed worker w_i we define a choice map C_{w_i} on $\{w_i\} \times F \times D$ by applying Theorem 3.9 with Y, q, Y_n, q_n and the preference relations on Y given as follows (for brevity we write (i,j,k) instead of (w_i, f_j, d_k) in the preference relations):

$$\begin{aligned} &Y := \{w_i\} \times \{f_1, f_2, f_3\} \times \{d_1, d_2\}, \quad q = 6, \\ &Y_k := \{w_i\} \times \{f_1, f_2, f_3\} \times \{d_k\}, \quad q_k = 1, \quad k = 1, 2, \\ &(i,i,1) \succ (i,i,2) \succ (i,i+1,1) \succ (i,i+1,2) \succ (i,i+2,1) \succ (i,i+2,2). \end{aligned}$$

We note that the quota $q = 6$ is ineffective.

Step 2. Applying Proposition 3.12 we combine the three choice maps of the preceding step into a global choice map C_W on $W \times F \times D$ by setting

$$C_W(A) := \bigcup_{i=1}^{3} C_{w_i} \left(A \cap (\{w_i\} \times F \times D) \right)$$

for every $A \subset W \times F \times D$.

Step 3. For each firm f_j we define a choice map C_{f_j} on $W \times \{f_j\} \times D$ by applying Theorem 3.9 with Y, q, Y_n, q_n and the preference relations on Y given as follows, still writing (i,j,k) instead of (w_i, f_j, d_k) for brevity:

$$\begin{aligned} &Y := \{w_1, w_2, w_3\} \times \{f_j\} \times \{d_1, d_2\}, \quad q = 2, \\ &Y_i := \{w_i\} \times \{f_j\} \times \{d_1, d_2\}, \quad q_i = 1, \quad i = 1, 2, 3, \\ &(j+1,j,2) \succ (j+1,j,1) \succ (j+2,j,2) \succ (j+2,j,1) \succ (j,j,2) \succ (j,j,1). \end{aligned}$$

Step 4. Applying Proposition 3.12 we combine the three choice maps of the preceding step into a global choice map C_F on $W \times F \times D$ by setting

$$C_F(A) := \bigcup_{j=1}^{3} C_{f_j} \left(A \cap (W \times \{f_j\} \times D) \right)$$

for every $A \subset W \times F \times D$.

Step 5. Thanks to Theorem 3.9 and Proposition 3.12 the choice maps C_W and C_F satisfy the hypotheses of Theorem 3.6. We apply the algorithm as described in Remark 3.7 by starting with $X_0 := X$ and computing Y_1, X_2, Y_3, X_4 by the formulas

$$Y_{n+1} := (X \setminus X_n) \cup C_W(X_n) \quad \text{and} \quad X_{n+1} := (X \setminus Y_n) \cup C_F(Y_n).$$

We obtain that $X_2 = X_4$ and therefore $S = C_W(X_2)$. The results are summarized in the following table:

w_i	f_j	d_k	X_0	Y_1	X_2	Y_3	X_4	S
1	1	1	x	x		x		
1	1	2	x	x	x	x	x	x
1	2	1	x		x	x	x	x
1	2	2	x		x		x	
1	3	1	x		x		x	
1	3	2	x		x		x	
2	1	1	x		x		x	
2	1	2	x		x		x	
2	2	1	x	x		x		
2	2	2	x	x	x	x	x	x
2	3	1	x		x	x	x	x
2	3	2	x		x		x	
3	1	1	x		x	x	x	x
3	1	2	x		x		x	
3	2	1	x		x		x	
3	2	2	x		x		x	
3	3	1	x	x		x		
3	3	2	x	x	x	x	x	x

In this *worker-optimal* solution each worker is hired by the second most preferred firm for the first day and by the most preferred firm for the second day.

Step 6. Applying the algorithm of Remark 3.7 by starting with $Y_0 := X$ and computing X_1, Y_2, X_3, Y_4 by the above formulas we obtain that $Y_2 = Y_4$ and therefore $S = C_F(Y_2)$. The results are summarized in the following table:

w_i	f_j	d_k	Y_0	X_1	Y_2	X_3	Y_4	S
1	1	1	x		x		x	
1	1	2	x		x		x	
1	2	1	x		x		x	
1	2	2	x	x	x	x	x	x
1	3	1	x		x	x	x	x
1	3	2	x	x		x		
2	1	1	x		x	x	x	x
2	1	2	x	x		x		
2	2	1	x		x		x	
2	2	2	x		x		x	
2	3	1	x		x		x	
2	3	2	x	x	x	x	x	x
3	1	1	x		x		x	
3	1	2	x	x	x	x	x	x
3	2	1	x		x	x	x	x
3	2	2	x	x		x		
3	3	1	x		x		x	
3	3	2	x		x		x	

In this *firm-optimal* solution each firm hires the most preferred worker for the first day and by the second most preferred worker for the second day.

Remark 4.1. The stable schedule matchings as studied by Baiou and Balinski [3] and Alkan and Gale [2] enter the present framework as a special case. For simplicity we consider the discrete case, when a contract is not only a worker-firm pair (w_i, f_j), but it also contains the number of working hours. We denote by $D = \{1, 2, \ldots\}$ the set of possible numbers of working hours with k meaning the kth working hour. For each worker w_i, if there is a preference ranking $f_{j_1} \succ f_{j_2} \succ \cdots$ among the firms, then we extend it to the preference ranking

$$\begin{aligned} &(i, j_1, 1) \succ (i, j_1, 2) \succ \cdots \succ (i, j_1, q^w_{i,j_1}) \\ \succ &(i, j_2, 1) \succ (i, j_2, 2) \succ \cdots \succ (i, j_2, q^w_{i,j_2}) \\ \succ &\cdots \\ &\vdots \end{aligned}$$

where $q^w_{i,j}$ denotes the maximum number of working hours accepted by worker w_i in firm f_j. Similarly, for each firm f_j, if there is a preference ranking $w_{i_1} \succ w_{i_2} \succ \cdots$ among the workers, then we extend it to the preference ranking

$$\begin{aligned} &(i_1, j, 1) \succ (i_1, j, 2) \succ \cdots \succ (i_1, j, q^f_{i_1,j}) \\ \succ &(i_2, j, 1) \succ (i_2, j, 2) \succ \cdots \succ (i_2, j, q^f_{i_2,j}) \\ \succ &\cdots \\ &\vdots \end{aligned}$$

where $q^f_{i,j}$ denotes the maximum number of working hours accepted by firm f_j for worker w_i. Once a stable set S found, the number of working hours of worker w_i in firm f_j is the biggest integer k such that $(i, j, k) \in S$.

Acknowledgement. The authors thank the referee for several useful comments, and to pointing out a similarity between our proof of Theorem 3.9 and the matroid greedy algorithm.

References

[1] H. Adachi, *On a characterization of stable matchings*, Econom. Lett. 68 (2000), 1, 43–49.

[2] A. Alkan, D. Gale, *Stable schedule matching under revealed preference*, J. Economic Theory 112 (2003), 289–306.

[3] M. Baiou, M. Balinski, *The stable allocation (or ordinal transportation) problem*, Math. Oper. Res. 27 (2002), 485–503.

[4] C. Blair, *The lattice structure of the set of stable matchings with multiple partners,* Mathematics of Operations Research 13 (1988), 619–628.

[5] Alexander S. Kelso jun., V. P. Crawford, *Job matching, coalition formation, and gross substitutes* , Econometrica 50 (1982), 1483–1504.

[6] F. Echenique, J. Ovideo, *Core many-to-one matchings by fixed-point methods,* Journal of Economic Theory 115 (2004), 358–376.

[7] F. Echenique, J. Ovideo, *A theory of stability in many-to-many matching markets,* Theoretical Economics 1 (2006), 233–273.

[8] T. Feder, *A new fixed point approach for stable networks and stable marriages*, Twenty-first Symposium on the Theory of Computing (Seattle, WA, 1989). J. Comput. System Sci. 45 (1992), 2, 233–284.

[9] T. Fleiner, *A matroid generalization of the stable matching polytope*, Integer programming and combinatorial optimization (Utrecht, 2001), 105–114, Lecture Notes in Comput. Sci., 2081, Springer, Berlin, 2001.
[10] T. Fleiner, *Stable matchings through fixed points and graphs*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 51 (2008), 69–116 (2009).
[11] T. Fleiner, *A fixed-point approach to stable matchings and some applications*, Math. Oper. Res. 28 (2003), 103–126.
[12] D. Gale, L. S. Shapley, *College admissions and the stability of marriage*, Amer. Math. Monthly 69 (1962), 1, 9–15.
[13] J. W. Hatfield, P. Milgrom, *Matching with contracts*, American Economic Review 95 (2005), 4, 913–935.
[14] B. Klaus, M. Walzl, *Stable many-to-many matchings with contracts*, Mathematics of Operations Research, 45 (2009) 422–434.
[15] B. Knaster, *Un théorème sur les fonctions d'ensembles*, Ann. Soc. Polon. Math. 6 (1928), 133–134.
[16] A. E. Roth, *Stability and polarization of interests in job matching*, Econometrica 52 (1984), 1, 47–57.
[17] A. E. Roth, M. Sotomayor, *Two-sided matching. A study in game-theoretic modeling and analysis*, Econometric Society Monographs, 18. Cambridge University Press, Cambridge, 1990.
[18] A. Subramanian, *A new approach to stable matching problems*, SIAM J. Comput. 23 (1994), 4, 671–700.
[19] A. Tarski, *Quelques théorèmes généraux sur les images d'ensembles*, Ann. Soc. Polon. Math. 6 (1928), 132–133.
[20] A. Tarski, *A lattice-theoretical fixpoint theorem and its applications*, Pacific J. Math. 5 (1955), 285–310.

Département de mathématique, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg Cedex, France
E-mail address: komornik@math.unistra.fr

UFR de mathématique et d'informatique, Université de Strasbourg, 7 rue René Descartes, 67084 Strasbourg Cedex, France
E-mail address: z.komornik@gmail.com

University of Maryland Baltimore County, Department of Economics, 1000 Hilltop Circle, Baltimore, MD 21250, USA
E-mail address: ckviauro@umbc.edu